Exercise 3.4.12

Solve the following *nonhomogeneous* problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L} \quad [\text{assume that } 2 \neq k(3\pi/L)^2]$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Use the following method. Look for the solution as a Fourier cosine series. Justify all differentiations of infinite series (assume appropriate continuity).

Solution

In order for the homogeneous Neumann boundary conditions to be satisfied, we assume the solution has the form of a Fourier cosine series.

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\frac{n\pi x}{L} \tag{1}$$

Since u is continuous, this is justified. Apply the initial condition now to determine $A_0(0)$ and $A_n(0)$.

$$u(x,0) = A_0(0) + \sum_{n=1}^{\infty} A_n(0)\cos\frac{n\pi x}{L} = f(x) \quad \rightarrow \quad \begin{cases} A_0(0) = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx \\ A_n(0) = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \end{cases}$$

These formulas for $A_0(0)$ and $A_n(0)$ will be needed later. Assuming $\partial u/\partial t$ is continuous, term-by-term differentiation with respect to t is valid.

$$\frac{\partial u}{\partial t} = A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L}$$

Because u is continuous, differentiation of the cosine series with respect to x is valid.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty}\left(-\frac{n\pi}{L}\right)A_n(t)\sin\frac{n\pi x}{L}$$

$\partial u/\partial x$ is continuous and $u_x(0,t) = u_x(L,t) = 0$, so differentiation of this sine series with respect to x is valid.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right)A_n(t)\cos\frac{n\pi x}{L}$$

Substitute these infinite series into the PDE.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L} = k\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right)A_n(t)\cos\frac{n\pi x}{L} + e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}$$

Bring them both to the left side.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L} + k\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{L^2}\right)A_n(t)\cos\frac{n\pi x}{L} = e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}$$

Combine them and factor the summand.

$$A_0'(t) + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L} = e^{-t} + e^{-2t}\cos\frac{3\pi x}{L} \tag{2}$$

To obtain $A_0(t)$, integrate both sides with respect to x from 0 to L.

$$\int_0^L \left\{A_0'(t) + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L}\right\}dx = \int_0^L \left(e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}\right)dx$$

Split up the integral on both sides and bring the constants in front.

$$A_0'(t)\int_0^L dx + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\underbrace{\int_0^L \cos\frac{n\pi x}{L}\,dx}_{=\,0} = e^{-t}\int_0^L dx + e^{-2t}\underbrace{\int_0^L \cos\frac{3\pi x}{L}\,dx}_{=\,0}$$

Evaluate the integrals.

$$A_0'(t)(L) = e^{-t}(L)$$

Divide both sides by L.

$$A_0'(t) = e^{-t}$$

Integrate both sides with respect to t.

$$A_0(t) = -e^{-t} + C_1$$

Apply the initial condition found in the beginning to determine C_1.

$$A_0(0) = -1 + C_1 = \frac{1}{L}\int_0^L f(x)\,dx \quad \rightarrow \quad C_1 = 1 + \frac{1}{L}\int_0^L f(x)\,dx$$

As a result,

$$A_0(t) = 1 - e^{-t} + \frac{1}{L}\int_0^L f(x)\,dx.$$

To get $A_n(t)$, multiply both sides of equation (2) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$A_0'(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} = e^{-t}\cos\frac{p\pi x}{L} + e^{-2t}\cos\frac{3\pi x}{L}\cos\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left\{A_0'(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right\}dx$$
$$= \int_0^L \left(e^{-t}\cos\frac{p\pi x}{L} + e^{-2t}\cos\frac{3\pi x}{L}\cos\frac{p\pi x}{L}\right)dx$$

Split up the integrals and bring the constants in front.

$$A_0'(t)\underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=0}+\sum_{n=1}^{\infty}\left[A_n'(t)+\frac{kn^2\pi^2}{L^2}A_n(t)\right]\int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx$$

$$=e^{-t}\underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=0}+e^{-2t}\int_0^L \cos\frac{3\pi x}{L}\cos\frac{p\pi x}{L}\,dx$$

The cosine functions are orthogonal, so the second integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\left[A_n'(t)+\frac{kn^2\pi^2}{L^2}A_n(t)\right]\int_0^L \cos^2\frac{n\pi x}{L}\,dx = e^{-2t}\int_0^L \cos\frac{3\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

In addition, the integral on the right is zero if $n \neq 3$, and it's nonzero if $n = 3$.

$$\left[A_n'(t)+\frac{kn^2\pi^2}{L^2}A_n(t)\right]\int_0^L \cos^2\frac{n\pi x}{L}\,dx = \begin{cases} e^{-2t}\int_0^L \cos^2\frac{3\pi x}{L}\,dx & n=3 \\ 0 & n\neq 3\end{cases}$$

Evaluate the integrals.

$$\left[A_n'(t)+\frac{kn^2\pi^2}{L^2}A_n(t)\right]\left(\frac{L}{2}\right) = \begin{cases} e^{-2t}\left(\frac{L}{2}\right) & n=3 \\ 0 & n\neq 3\end{cases}$$

Divide both sides by $L/2$.

$$A_n'(t)+\frac{kn^2\pi^2}{L^2}A_n(t) = \begin{cases} e^{-2t} & n=3 \\ 0 & n\neq 3\end{cases}$$

This is a first-order linear ODE, so it can be solved with an integrating factor I.

$$I=\exp\left(\int^t \frac{kn^2\pi^2}{L^2}\,ds\right)=\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Multiply both sides of the ODE by I.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n'(t)+\frac{kn^2\pi^2}{L^2}\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t) = \begin{cases} e^{-2t}\exp\left(\frac{kn^2\pi^2}{L^2}t\right) & n=3 \\ 0 & n\neq 3\end{cases}$$

The left side can be written as $d/dt(IA_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t)\right] = \begin{cases} \exp\left[\left(\frac{kn^2\pi^2}{L^2}-2\right)t\right] & n=3 \\ 0 & n\neq 3\end{cases}$$

Integrate both sides with respect to t.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t) = \begin{cases} \dfrac{1}{\frac{kn^2\pi^2}{L^2}-2}\exp\left[\left(\dfrac{kn^2\pi^2}{L^2}-2\right)t\right]+C_2 & n=3 \\ C_3 & n\neq 3 \end{cases}$$

Solve for $A_n(t)$.

$$A_n(t) = \begin{cases} \dfrac{1}{\frac{kn^2\pi^2}{L^2}-2}e^{-2t}+C_2\exp\left(-\dfrac{kn^2\pi^2}{L^2}t\right) & n=3 \\ C_3\exp\left(-\frac{kn^2\pi^2}{L^2}t\right) & n\neq 3 \end{cases}$$

Apply the initial condition to determine C_2 and C_3.

$$A_n(0) = \begin{cases} \dfrac{1}{\frac{kn^2\pi^2}{L^2}-2}+C_2 & n=3 \\ C_3 & n\neq 3 \end{cases} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \rightarrow \quad \begin{cases} C_2 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{3\pi x}{L}\,dx - \dfrac{1}{\frac{9k\pi^2}{L^2}-2} \\ C_3 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \end{cases}$$

Therefore,

$$\begin{aligned} A_n(t) &= \begin{cases} \dfrac{1}{\frac{kn^2\pi^2}{L^2}-2}e^{-2t}+\left[\dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{3\pi x}{L}\,dx - \dfrac{1}{\frac{9k\pi^2}{L^2}-2}\right]\exp\left(-\dfrac{kn^2\pi^2}{L^2}t\right) & n=3 \\ \left[\dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(-\dfrac{kn^2\pi^2}{L^2}t\right) & n\neq 3 \end{cases} \\ &= \begin{cases} \dfrac{1}{\frac{9k\pi^2}{L^2}-2}\left[e^{-2t}-\exp\left(-\dfrac{9k\pi^2}{L^2}t\right)\right]+\left[\dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{3\pi x}{L}\,dx\right]\exp\left(-\dfrac{9k\pi^2}{L^2}t\right) & n=3 \\ \left[\dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(-\dfrac{kn^2\pi^2}{L^2}t\right) & n\neq 3 \end{cases} \end{aligned}$$

and the solution to the PDE is

$$\begin{aligned} u(x,t) &= A_0(t)+\sum_{n=1}^{\infty}A_n(t)\cos\frac{n\pi x}{L} \\ &= 1-e^{-t}+\left[\frac{1}{L}\int_0^L f(x)\,dx\right]+\sum_{\substack{n=1\\n\neq 3}}^{\infty}\left[\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\cos\frac{n\pi x}{L} \\ &\quad + \left\{\frac{1}{\frac{9k\pi^2}{L^2}-2}\left[e^{-2t}-\exp\left(-\frac{9k\pi^2}{L^2}t\right)\right]+\left[\frac{2}{L}\int_0^L f(x)\cos\frac{3\pi x}{L}\,dx\right]\exp\left(-\frac{9k\pi^2}{L^2}t\right)\right\}\cos\frac{3\pi x}{L}. \end{aligned}$$

This answer is in disagreement with the one at the back of the book, specifically e^{-2t} instead of e^{2t}.